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Fuzzy complex projective spaces and their star-products

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Abstract

We derive an explicit expression for an associative $*$-product on the fuzzy complex projective space, $\mathbf{CP}_F^{N-1}$. This generalises previous results for the fuzzy 2-sphere and gives a discrete non-commutative algebra of functions on $\mathbf{CP}_F^{N-1}$, represented by matrix multiplication. The matrices are restricted to ones whose dimension is that of the totally symmetric representations of $SU(N)$. In the limit of infinite-dimensional matrices we recover the commutative algebra of functions on $\mathbf{CP}^{N-1}$. Derivatives on $\mathbf{CP}_F^{N-1}$ are also expressed as matrix commutators.

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1. Introduction

The concept of non-commutative geometry [1,2] is emerging as one of the most promising and interesting new tools in quantum field theory. It is also providing novel insights into the possible space–time structure at the level of quantum gravity. In quantum field theory it can provide a regularisation technique which is completely compatible with the space–time

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symmetries of the theory, [3–17], while in quantum gravity it points the way to radical approaches. It has also found several applications in string theory [18]. In its matrix model or 'fuzzy' form[2] it promises a radical alternative to lattice field theory, where problems such as chiral fermion doubling are readily avoided [13]. A major obstacle to the development of this fuzzy alternative to lattice theories is the paucity of fuzzy spaces with explicit descriptions.

An important ingredient in understanding the continuum limit of these fuzzy models is the $*$-product. This is a non-commutative product for functions that, in the case of fuzzy spaces, represents the matrix product. An explicit example of a $*$-product is known for the fuzzy 2-sphere [4]. It is known that a $*$-product can be defined as a formal power series on any manifold that admits a symplectic or Poisson structure [19,20], but few explicit examples are known.

In this paper we present an explicit construction of a $*$-product on the fuzzy complex projective space $\mathbf{CP}_F^{N-1}$. While a non-commutative $*$-product on the continuum $\mathbf{CP}^{N-1}$ is known, in an integral representation (see, e.g. [21]), to our knowledge this is the first time an expression for a $*$-product on the fuzzy $\mathbf{CP}_F^{N-1}$ has been given. The construction presented here is a generalisation of the construction of the $*$-product on the 2-sphere given in [4].

The layout of the paper is as follows: in the next section we give a brief discussion of harmonic expansions of functions on fuzzy spaces, by way of motivation for $*$-products and their use in quantum field theory; in Section 3 we give a general discussion of $*$-products analysing when they can be expected to exist and, in particular, when the given construction, based on equivariant products, should exist; Sections 4 and 5 give a description of $\mathbf{CP}^{N-1}$ in terms of global coordinates; in Section 6 the $*$-product on fuzzy $\mathbf{CP}_F^{N-1}$ is constructed in terms of projectors and Section 7 describes the relation between derivatives in the continuum and their discrete representation on $\mathbf{CP}_F^{N-1}$; finally Section 8 summarises the conclusions. Some technical results required for the main text are reserved for the appendices.

2. Fuzzy functions

If one attempts to discretise field theory on a continuous manifold there are immediate problems that have to be overcome. Not least is the fact that continuum symmetries are lost and great care must be exercised in ensuring that they are recovered again when the continuum limit is taken. Another problem, which occurs in Fourier space and is not often remarked upon because the resolution appears to be so simple, is that the algebra of functions in truncated Fourier space does not close in general. For example if one Fourier analyses functions on a circle,

$$f(\theta) = \sum_{n=-\infty}^{\infty} f_n \,\mathrm{e}^{\mathrm{i}n\theta}, \tag{2.1}$$

and approximates them by cutting off the Fourier series at some maximum frequency, L,

$$f_L(\theta) = \sum_{n=-L}^{L} f_n \,\mathrm{e}^{\mathrm{i}n\theta} \tag{2.2}$$

[2] Fuzzy spaces are discrete matrix approximations to continuous manifolds.

then the product of two such functions will in general extend to frequencies up to $2L$ and so the algebra of truncated functions does not close under multiplication. The same problem manifests itself when functions are expanded on a sphere in terms of spherical harmonics and then approximated simply by cutting off the expansion at some maximum angular momentum. An obvious naïve remedy is to project after multiplying and just throw away all the frequencies higher than L. While this brute force method may work, it is not without its problems—for example such a process is non-associative in general. There are sometimes situations where a more elegant method presents itself which at the same time does less violence to the group representation theory and allows certain spaces to be discretised while preserving their continuum symmetries. One approach is to identify the coefficients in an harmonic expansion with elements of a matrix. If the multiplication of two functions can be implemented by matrix multiplication then the matrix algebra will close and no projection is necessary.

Consider for example a two-dimensional sphere which can be written as the coset space $S^2 \cong SU(2)/U(1)$. A general function on $SU(2)$ can be expanded in terms of D-matrices,

$$f = \sum_{l=0,1/2,1,\dots}^{\infty} \sum_{m,m'=-l}^{l} f^l_{m,m'} D^l_{m,m'}. \tag{2.3}$$

To restrict this to a function on S^2 the expansion must be restricted to entries of the D-matrices (or linear combinations of them) which are invariant under the right action of $U(1)$. The only such entries have $m' = 0$, and hence have integral l, since m' is the $U(1)$ quantum number. The D-matrices can be constructed so that $D^l_{m,0}$ are independent of the third Euler angle on $SU(2)$, then they depend only on the polar and azimuthal angles on S^2 and they are essentially the spherical harmonics—in standard notation $D^l_{m,0} = \sqrt{4\pi/(2l+1)}(-1)^m Y^l_{-m}$. Now the representation theory of $SU(2)$ allows a re-arrangement of the coefficients in a truncated expansion

$$f_L(\theta,\phi) = \sum_{l=0}^{L} \sum_{m=-l}^{l} f^l_m D^l_{m,0}(\theta,\phi) \tag{2.4}$$

into a square matrix. For any given value of l, $\sum_{m=-l}^{m=l} f^l_m D^l_{m,0}$ is just one component of the row vector obtained from the right action of an element of $SU(2)$ on the row vector with components f^l_m, $m = -l, \dots, l$. For a fixed l the row vectors with components f^l_m carry an irreducible representation of $SU(2)$. The set of all coefficients in the expansion (2.4) therefore constitute a reducible representation. For example if $L = 1$ the number of coefficients f^l_m is $1 + 3 = 2 \times 2$, if $L = 2$ the number is $1 + 3 + 5 = 3 \times 3$ and so on. For general L, the number of terms in this expansion at each value of l is $2l + 1$ giving a total of

$$(L+1)^2 = 1 + 3 + 5 + \cdots + (2L+1) \tag{2.5}$$

coefficients, which are in the reducible $(L+1) \times (L+1)$ representation of $SU(2)$. Multiplication of two functions truncated at the same value of L can now be defined as multiplication of their associated $(L+1) \times (L+1)$ matrices and group representation theory ensures that the resulting product, being itself a $(L+1) \times (L+1)$ matrix, only entails angular momentum up to L. These matrices define the fuzzy sphere and this matrix multiplication induces the $*$-product on the fuzzy sphere. It is a non-commutative associative product and it will be shown later that it reduces to the familiar commutative product of functions in the limit $L \to \infty$.

The 2-sphere is rather special in that $SU(2)$ has irreducible representations for every integer and so matrices of any size can be used to approximate functions, but more general coset spaces are more restrictive. Consider, for example, $\mathbf{CP}^2 \cong SU(3)/U(2)$. Again a function on $SU(3)$ can be expanded in terms of the representation matrices of $SU(3)$,

$$f = \sum_{l_1,l_2} \sum_{I,I_3,Y;I',I'_3,Y'} f^{(l_1,l_2)}_{I,I_3,Y;I',I'_3,Y'} D^{(l_1,l_2)}_{I,I_3,Y;I',I'_3,Y'}, \tag{2.6}$$

where the integers l_1 and l_2 label the irreducible representations of $SU(3)$ (l_1 and l_2 are, respectively, the number of symmetric $\mathbf{3}$'s and the number of symmetric $\bar{\mathbf{3}}$'s in the Young diagram of the representation) and I, I_3 and Y are the isospin, third component of isospin and hypercharge, respectively, of the little group $U(2)$ (these can be used to label the weights of any irreducible representation of $SU(3)$ unambiguously). To describe a scalar function on $\mathbf{CP}^2$ we must pick out the parts of the $SU(3)$ representation matrices that are $U(2)$ singlets under right multiplication. This immediately eliminates all the complex representations of $SU(3)$: the $\mathbf{3}$, $\bar{\mathbf{3}}$, $\mathbf{6}$, $\bar{\mathbf{6}}$, etc. The remaining real representations require $l_1 = l_2 = l$ and have dimension $(l+1)^3$. Again of these only one column of each representation matrix survives—the one given by $I' = I'_3 = Y' = 0$. The column vectors $\mathcal{Y}^{(l,l)}_{I,I_3,Y} := D^{(l,l)}_{I,I_3,Y;0,0,0}$ thus constitute generalised spherical harmonics on $\mathbf{CP}^2$ and functions can be expanded as

$$f = \sum_{l} \sum_{I,I_3,Y} f^{(l,l)}_{I,I_3,Y} \mathcal{Y}^{(l,l)}_{I,I_3,Y}. \tag{2.7}$$

Again the coefficients fall into representations of $SU(3)$:

$$\mathbf{1} + \mathbf{8} + \mathbf{27} + \mathbf{64} + \cdots . \tag{2.8}$$

Truncating at some maximum value, L, of l always allows the number of coefficients to be arranged in a square matrix: thus $L = 1$ gives $\bar{\mathbf{3}} \times \mathbf{3} = \mathbf{1}+\mathbf{8}$; $L = 2$ gives $\bar{\mathbf{6}} \times \mathbf{6} = \mathbf{1}+\mathbf{8}+\mathbf{27}$; $L = 3$ gives $\overline{\mathbf{10}} \times \mathbf{10} = \mathbf{1} + \mathbf{8} + \mathbf{27} + \mathbf{64}$; and so on. Truncating at L results in square matrices of size $(L+2)(L+1)/2$, which is the dimension of the symmetric tensor product of L $\mathbf{3}$'s (or L $\bar{\mathbf{3}}$'s), and

$$\sum_{l=0}^{L} (l+1)^3 = \frac{(L+2)^2(L+1)^2}{4}. \tag{2.9}$$

Again the group representation theory ensures that matrix multiplication keeps within the same representations and never goes above L.

This construction generalises to the higher-dimensional complex projective spaces $\mathbf{CP}^{N-1}$ where the matrices at the smallest non-trivial approximation begin with $\bar{\mathbf{N}} \times \mathbf{N} = \mathbf{1} + (\mathbf{N}^2 - \mathbf{1})$, the next being $\overline{\mathbf{N}(\mathbf{N}+\mathbf{1})/\mathbf{2}} \times \mathbf{N}(\mathbf{N}+\mathbf{1})/\mathbf{2} = \mathbf{1} + (\mathbf{N}^2 - \mathbf{1}) + \mathbf{N}^2(\mathbf{N}^2 + \mathbf{2N} - \mathbf{3})/\mathbf{4}$, etc. Truncating at L gives a $[(N-1+L)!/(N-1)!L!] \times [(N-1+L)!/(N-1)!L!]$ matrix representation approximation of $\mathbf{CP}^{N-1}$. A similar truncation works for unitary Grassmannian manifolds [17]. However, it is not always the case that the representation theory allows the expansion of a function on a coset space to be described in terms of square matrices like this. When it can be done we can define a $*$-product on a fuzzy version of the space.

3. On $*$-products

In this section we present a general discussion of $*$-products with emphasis on "equivariant" $*$-products.

Suppose we have an algebra $\hat{\mathcal{A}}$ of linear operators on a finite-dimensional vector space. We assume that, if $\hat{F} \in \hat{A}$ then its Hermitian conjugate $\hat{F}^\dagger \in \hat{A}$, so that $\hat{A}$ is a $*$-algebra. Let a connected compact Lie group $\mathcal{G} = \{g\}$ act on $\hat{\mathcal{A}}$ by adjoint action of unitary matrices:

$$\hat{F} \mapsto \hat{D}(g)\hat{F}\hat{D}^{-1}(g), \qquad \hat{D}^\dagger(g)D(g) = \mathbf{1}. \tag{3.1}$$

We can assume, by Wedderburn's theorem, [22, Theorem 6.3.8], that $\hat{A}$ is the direct sum of full matrix algebras, Mat_d, of $d \times d$ matrices: $\hat{A} = \oplus_d \mathrm{Mat}_d$. As the $\hat{D}(g)$ action preserves $\hat{A}$, it also decomposes as $\hat{D}(g) = \oplus_d \hat{D}^{(d)}(g)$. Since Mat_d is simple, the two-sided ideals of $\hat{A}$ are all direct sums of some of the Mat_d or just $\{0\}$.

To get a $*$-product we need, in addition, a function $\hat{\rho}^*$ on a manifold $\mathcal{M}$ with values in $\hat{A}^*$, the dual of $\hat{A}$. Then, $\langle \hat{\rho}^*, \hat{F} \rangle := F$ is a function on $\mathcal{M}$:

$$\langle \hat{\rho}^*, \hat{F} \rangle(\xi) \equiv \langle \hat{\rho}^*(\xi), \hat{F} \rangle = F(\xi), \tag{3.2}$$

where $\xi \in \mathcal{M}$. This map $\hat{\mathcal{A}} \to \mathcal{C}_F^\infty(\mathcal{M}) \subset \mathcal{C}^\infty(\mathcal{M})$ (assuming appropriate continuity requirements) induces an algebra structure on $\mathcal{C}_F^\infty(\mathcal{M})$ if its kernel, Ker, is a two-sided ideal in $\hat{\mathcal{A}}$, that is if Ker is a direct sum of some of the Mat_d or $\{0\}$. If that is the case, $\mathcal{C}_F^\infty(\mathcal{M}) \cong \hat{\mathcal{A}}/\mathrm{Ker}$, and its algebra product is defined by

$$(F * G)(\xi) = \langle \hat{\rho}^*(\xi), \hat{F}\hat{G} \rangle, \tag{3.3}$$

where $\hat{F}, \hat{G} \in \hat{\mathcal{A}}$.

The action (3.1) on $\hat{\mathcal{A}}$ induces an action on its dual $\hat{\mathcal{A}}^*$ which we denote by $\hat{F}^* \to \hat{D}^*(g)^{-1}\hat{F}^*\hat{D}^*(g)$:

$$\langle \hat{F}^*, \hat{D}(g)\hat{F}\hat{D}(g)^{-1} \rangle = \langle \hat{D}(g)^{*-1}\hat{F}^*\hat{D}(g)^*, \hat{F} \rangle. \tag{3.4}$$

Until now there is no requirement that $\hat{\rho}^*(\xi)$ is a state or has equivariance. The setting is very general. Suppose we now ask that $\hat{\rho}(\xi)$ is a state:

$$\langle \hat{\rho}^*(\xi), \hat{F}^\dagger \rangle = \overline{\langle \hat{\rho}^*(\xi), \hat{F} \rangle}, \tag{3.5}$$

$$\langle \hat{\rho}^*(\xi), \hat{F}^\dagger \hat{F} \rangle \geq 0, \tag{3.6}$$

$$\langle \hat{\rho}^*(\xi), \hat{\mathbf{1}} \rangle = 1, \tag{3.7}$$

where bar denotes complex conjugation. Then $\hat{\rho}^*(\xi)$ can be identified with a density matrix $\hat{\rho}(\xi)$ by setting

$$\langle \hat{\rho}^*(\xi), \hat{F} \rangle = \mathrm{Tr}(\hat{\rho}(\xi)\hat{F}). \tag{3.8}$$

For equivariance we assume that g acts transitively on $\mathcal{M}$, $\xi \to g\xi$, such that

$$\hat{\rho}^*(g\xi) = \hat{D}^*(g)\hat{\rho}^*(\xi)\hat{D}^*(g^{-1}). \tag{3.9}$$

Now each Mat_d and $\hat{\mathcal{A}}$ can be decomposed into irreducible tensor operators:

$$\mathrm{Mat}_d = \mathrm{Span}\{\hat{T}^{(l)}_M(d)\}, \qquad \hat{D}^{(d)}(g)\hat{T}^{(l)}_M(d)\hat{D}^{(d)}(g)^{-1} = \sum_{M'} \hat{T}^{(l)}_{M'}(d)D^{(l)}_{M'M}(g), \tag{3.10}$$

where $g \mapsto D^{(l)}(g)$ is a unitary irreducible representation. Let $\{\hat{T}^{*(l)}_M(d)\}$ be the dual basis:

$$\langle \hat{T}^{*(l')}_{M'}(d'), \hat{T}^{(l)}_M(d)\rangle = \delta_{ll'}\delta_{dd'}\delta_{MM'}. \tag{3.11}$$

It transforms as

$$\hat{D}^{*(d)}(g^{-1})\hat{T}^{*(l)}_M(d)\hat{D}^{*(d)}(g) = \sum_{M'} \hat{T}^{*(l)}_{M'}(d)D^{*(l)}_{M'M}(g^{-1}), \tag{3.12}$$

where

$$\hat{D}^{*(l)}_{M'M}(g)\hat{D}^{(l)}_{M'N}(g) = \delta_{MN}. \tag{3.13}$$

We can expand

$$\hat{\rho}^* = \sum_{d,l,M} \rho^{(l,d)}_M \hat{T}^{*(l)}_M(d) := \sum_{l,d} \hat{\rho}^{*(l,d)},$$

where

$$\rho^{(l,d)}_M : \mathcal{M} \to \mathbf{C} \quad \text{and} \quad \hat{\rho}^{*(l,d)} = \sum_M \rho^{(l,d)}_M \hat{T}^{*(l)}_M(d). \tag{3.14}$$

Now Wedderburn's theorem implies that for a $*$-product to exist, either *all* functions $\rho^{(l,d)}_M$ for a fixed d, or *none*, must be zero, because if $\hat{\rho}^*$ has a kernel consistency requires that it be a full matrix algebra. In fact, because of equivariance, we shall now show that it is sufficient to check if $\hat{\rho}^{*(l,d)}$ is zero or not at one point, which we shall call the origin and denote by ξ_0. We have

$$\hat{\rho}^{*(l,d)}(g\xi_0) = \hat{D}^{*(d)}(g)\hat{\rho}^{*(l,d)}(\xi_0)\hat{D}^{*(d)}(g^{-1}), \tag{3.15}$$

or

$$\sum_M \rho^{(l,d)}_M(g\xi_0)\hat{T}^{*(l)}_M(d) = \sum_{M,M'} \rho^{(l,d)}_M(\xi_0)\hat{T}^{*(l)}_{M'}(d)D^{(l)}_{M'M}(g)$$
$$\Rightarrow \rho^{(l,d)}_{M'}(g\xi_0) = D^{(l)}_{M'M}(g)\rho^{(l,d)}_M(\xi_0). \tag{3.16}$$

So, from equivariance,

$$\hat{\rho}^{*(l,d)}(\xi_0) = 0 \Rightarrow \rho^{(l,d)}_M(\xi_0) = 0 \quad \text{and} \quad \hat{\rho}^{*(l,d)} = 0. \tag{3.17}$$

Thus, with equivariance, it is enough to check that $\hat{\rho}^{*(l,d)}(\xi_0) = 0$, either for all l or no l, for each fixed d, to verify if $*$ exists.

We remark that it is not necessary to assume (3.7) separately, as we can arrange to have it with (3.5) and (3.6):

$$\langle \hat{\rho}^*(g\xi_0), \hat{\mathbf{1}} \rangle = \langle \hat{\rho}^*(\xi_0), \hat{D}(g)\hat{\mathbf{1}}\hat{D}(g^{-1}) \rangle = \langle \hat{\rho}^*(\xi_0), \hat{\mathbf{1}} \rangle = \langle \hat{\rho}^*(\xi_0), \hat{\mathbf{1}}^\dagger \hat{\mathbf{1}} \rangle, \tag{3.18}$$

which, by (3.6), is a constant non-negative number, c. As the ideal containing $\hat{\mathbf{1}}$ is $\hat{\mathcal{A}}$, c cannot be zero if there is a non-trivial $*$-product. So we can work with $\hat{\rho}^*/c$ instead so that (3.7) is enforced. As for (3.5) and (3.6), they are natural. Eq. (3.5) gives real functions for Hermitian operators, and (3.6) gives $\bar{F} * F(\xi) \geq 0$.

Note that if functions on $\mathcal{M}$ do not carry an IRR l with the correct multiplicity, it can happen that $\hat{\mathcal{A}}$ admits no $*$-product. This problem occurs, for example, if $\hat{\mathcal{A}}$ is the 8×8 matrix algebra and $\hat{\rho}(\xi_0)$ is $a\mathbf{1} + b[\mathrm{ad}(Y)]$ (where $\mathrm{ad}(Y)$ is the adjoint generator of hypercharge) with a and b chosen so that (3.5) and (3.6) are satisfied. Then, $\hat{\rho}$ gives a map to functions on $\mathbf{CP}^2$. The latter has 8 only once, but $\hat{\mathcal{A}}$ has two 8's, so there is no $*$-product (for a general discussion see [23]). A $*$-product does, however, exist on $\mathbf{CP}^2$, for suitable $\hat{\rho}$, which we construct later.

It is useful to note the following. Quite generally, in the equivariant case, with t_A an orthonormal basis (in the trace norm) for the Lie algebra of G,

$$\mathrm{Tr}(\hat{\rho}(g\xi_0)t_A) = \xi_A(g) = (\mathrm{Ad}\, g)_{AB}\xi_B(1). \tag{3.19}$$

Writing $\hat{\rho}(g) = (\sum \eta_B(g)t_B +$ terms orthogonal to t_B), only the first term survives the tracing, so that $\eta_A = \xi_A$, with t_A normalised appropriately. ξ_A maps G/H to an adjoint orbit and provides coordinate functions on G/H.

To escape the limitation of only getting $*$-star products on adjoint orbits, we may have to modify the requirement of equivariance.

In the subsequent construction of a $*$-product on $\mathbf{CP}^{N-1}$ we shall restrict our considerations to the case where $\hat{\rho}$ is a rank 1 projector and we shall use the notation $\mathcal{P}$ for $\hat{\rho}$ (or $\mathcal{P}_L = \hat{\rho}$ for its L-fold symmetric product, as explained later).

4. Global coordinates on $\mathbf{CP}^{N-1}$

We now turn to an explicit construction of the complex projective space $\mathbf{CP}^{N-1}$, which can be defined as the space of vectors of unit norm in $\mathbf{C}^N$ modulo the phase. Since a unit vector $|\psi\rangle$ up to a phase defines a projection operator $\mathcal{P} := |\psi\rangle\langle\psi|$, an equivalent definition for $\mathbf{CP}^{N-1}$ is as the space of all projection operators of rank 1 on $\mathbf{C}^N$, i.e.,

$$\mathbf{CP}^{N-1} := \{\mathcal{P} \in \mathrm{Mat}_N; \mathcal{P}^\dagger = \mathcal{P}, \mathcal{P}^2 = \mathcal{P}, \mathrm{Tr}\,\mathcal{P} = 1\}. \tag{4.1}$$

To construct a set of global coordinates for $\mathbf{CP}^{N-1}$, we choose a basis for $N\times N$ Hermitian matrices $\{t_{\hat{A}}\}$, $\hat{A} = 0, \ldots, N^2-1$, consisting of $t_0 = \mathbf{1}/\sqrt{N}$ and $\{t_A\}$, $A = 1, \ldots, N^2-1$, forming an orthogonal basis for the Lie algebra of $SU(N)$. We will normalise them by requiring

$$\mathrm{Tr}(t_{\hat{A}}t_{\hat{B}}) = \delta_{\hat{A}\hat{B}}. \tag{4.2}$$

This requirement implies that $t_0 = (1/\sqrt{N})\mathbf{1}$ and t_A's are related to the Gell–Mann matrices λ_A by $t_A = \lambda_A/\sqrt{2}$. Thus,

$$t_A t_B = \frac{1}{N}\delta_{AB}\mathbf{1} + \frac{1}{\sqrt{2}}(d_{AB}^C + \mathrm{i} f_{AB}^C)t_C, \tag{4.3}$$

where f_{AB}^C and d_{AB}^C are, respectively, the structure constants and the components of the symmetric invariant tensor of $SU(N)$ in the Gell–Mann basis. The d-tensor is traceless on each pair of indices. For raising and lowering indices we will use the Kronecker δ.

Expanding $\mathcal{P}$ in terms of the basis,

$$\mathcal{P} = \xi^{\hat{A}} t_{\hat{A}} = \xi^0 t_0 + \xi^A t_A. \tag{4.4}$$

The condition that $\mathcal{P}$ is a rank 1 projection operator leads to the following conditions on the coordinates $\xi^{\hat{A}}$,

$$\xi^0 = \frac{1}{\sqrt{N}}, \qquad \xi^A \xi_A = \frac{N-1}{N}, \qquad d_{AB}^C \xi^A \xi^B = \frac{\sqrt{2}(N-2)}{N}\xi^C. \tag{4.5}$$

These form a set of quadratic constraints which describe $\mathbf{CP}^{N-1}$ embedded in the N^2-dimensional Euclidean space $\mathbf{R}^{N^2}$, or in $\mathbf{R}^{N^2-1}$ since ξ^0 is a fixed constant. For example, for $N = 2$ we have $A, B = 1, 2, 3$ and the above equations reduce to that of a sphere, or $\mathbf{CP}^1$, of radius $1/\sqrt{2}$ embedded in $\mathbf{R}^3$ because the d-tensor vanishes for $SU(2)$.

The coordinates for $\mathbf{CP}^{N-1}$ can be constructed easily by noting that any $\mathcal{P} \in \mathbf{CP}^{N-1}$ can be obtained from an arbitrarily chosen origin $\mathcal{P}_o$ by rotating it with $g \in U(N)$, $\mathcal{P} = g\mathcal{P}_o g^\dagger$. Of course there is no unique element g associated with $\mathcal{P}$. In fact, any two elements of $U(N)$ that are related by $g' = gh$, where $h \in U(1) \times U(N-1)$ give rise to the same point of $\mathbf{CP}^{N-1}$, as can be seen by going to the basis of $\mathbf{C}^N$ in which P_o is diagonal. (This leads to still another characterisation of the complex projective space, i.e., $\mathbf{CP}^{N-1} = U(N)/[U(1) \times U(N-1)]$.) Using this fact one can obtain coordinates, $\xi^{\hat{A}}$, corresponding to an arbitrary point of $\mathbf{CP}^{N-1}$, $\mathcal{P} = g\mathcal{P}_o g^\dagger$, from the coordinates $\xi_o^{\hat{A}}$ of the origin $\mathcal{P}_o$ as follows:

$$\xi^{\hat{A}} = \mathrm{Tr}(\mathcal{P} t^{\hat{A}}) = \mathrm{Tr}(g\mathcal{P}_o g^\dagger t^{\hat{A}}) = \xi_o^{\hat{B}}\, \mathrm{Tr}(g t_{\hat{B}} g^\dagger t^{\hat{A}}) = (\mathrm{Ad}(g))^{\hat{A}}_{\hat{B}} \xi_o^{\hat{B}}, \tag{4.6}$$

so that $\xi^{\hat{A}}$ map $\mathbf{CP}^{N-1}$ to an adjoint orbit of $U(N)$ fulfilling (3.19).

It is important for what follows that $\mathcal{P}$ fulfils the property (3.9):

$$g^{-1}\xi^A t_A g = (\mathrm{Ad}(g))_{AB}\xi_B t_A. \tag{4.7}$$

Here $g \in U(N)$ and $g \to \mathrm{Ad}(g)$ defines its adjoint representation.

5. The geometry of $\mathbf{CP}^{N-1}$

The coordinates $\xi^{\hat{A}}$, $\hat{A} = 0, \ldots, N^2 - 1$ constitute an over-complete, but globally well-defined, coordinate system for $\mathbf{CP}^{N-1}$. It is therefore useful to use them to describe geometrical structures on $\mathbf{CP}^{N-1}$ such as the Fubini–Study metric and Kähler structure. To this end let us regard $\mathbf{CP}^{N-1}$ as a manifold embedded in the space $\mathbf{R}^{N^2}$ of all Hermitian $N \times N$ matrices. At a given point $\mathcal{P} \in \mathbf{CP}^{N-1}$ we can decompose $\mathbf{R}^{N^2}$ into the subspace $\mathrm{T}_{\mathcal{P}}\mathbf{CP}^{N-1}$ consisting of vectors tangential to $\mathbf{CP}^{N-1}$ and its orthogonal complement. Since the action of $U(N)$ spans all directions tangential to $\mathbf{CP}^{N-1}$ at $\mathcal{P}$, and $\mathcal{P}$ is rotated by the adjoint action of $U(N)$, any vector in $\mathrm{T}_{\mathcal{P}}\mathbf{CP}^{N-1}$ must be of the form,

$$T = \mathrm{i}\,\mathrm{Ad}(\Lambda)\mathcal{P} = \mathrm{i}[\Lambda, \mathcal{P}] \tag{5.1}$$

for some Hermitian matrix Λ. This immediately implies that $\mathcal{T}$ must satisfy

$$\mathcal{T}^{\dagger} = \mathcal{T}, \qquad \{\mathcal{P}, \mathcal{T}\} = \mathcal{T}, \qquad \mathrm{Tr}\,\mathcal{T} = 0. \tag{5.2}$$

Note that if Λ is a generator of the stability subgroup $U(1) \times U(N-1)$, the RHS of (5.1) vanishes so that vectors $\mathcal{T}$ span a vector space of dimension of $N^2-(N-1)^2-1 = 2N-2$, which agrees with the dimension of $\mathbf{CP}^{N-1}$.

The vectors in the orthogonal complement of $\mathrm{T}_{\mathcal{P}}\mathbf{CP}^{N-1}$, on the other hand, can be represented by the generators $\mathcal{N}$ of the stability subgroup of $U(N)$. They satisfy

$$[\mathcal{P}, \mathcal{N}] = 0. \tag{5.3}$$

One can see this by noting that all such vectors are orthogonal to $\mathcal{T} = \mathrm{i}[\Lambda, \mathcal{P}]$,

$$\mathrm{Tr}(\mathcal{N}\mathcal{T}) = \mathrm{i}\,\mathrm{Tr}(\mathcal{N}[\Lambda, \mathcal{P}]) = \mathrm{i}\,\mathrm{Tr}([\mathcal{P}, \mathcal{N}]\Lambda) = 0. \tag{5.4}$$

These facts are now used to describe the Kähler structure on $\mathbf{CP}^{N-1}$. The Kähler structure consists of the following three mutually compatible structures:

1) *Complex structure*: For any Hermitian matrix $\mathcal{M}$, regarded as a vector at $\mathcal{P}$, define

$$J(\mathcal{M}) := -\mathrm{i}[\mathcal{P}, \mathcal{M}]. \tag{5.5}$$

If $\mathcal{M}$ is normal to $\mathbf{CP}^{N-1}$, i.e., if $\mathcal{M} = \mathcal{N}$, then $J(\mathcal{N}) = 0$ trivially. If $\mathcal{M}$ is tangential, i.e., $\mathcal{M} = \mathcal{T}$, then

$$\begin{aligned} J^2(\mathcal{T}) &= -[\mathcal{P}, [\mathcal{P}, \mathcal{T}]] = -\mathcal{P}(\mathcal{P}\mathcal{T} - \mathcal{T}\mathcal{P}) + (\mathcal{P}\mathcal{T} - \mathcal{T}\mathcal{P})\mathcal{P} \\ &= -\mathcal{P}\mathcal{T} - \mathcal{T}\mathcal{P} + 2\mathcal{P}\mathcal{T}\mathcal{P} = -\mathcal{T}, \end{aligned} \tag{5.6}$$

where in the last step we have used (5.2) and $\mathcal{P}\mathcal{T}\mathcal{P} = 0$ which follows immediately from that equation. Therefore, J is a complex structure on $\mathbf{CP}^{N-1}$. In view of (5.3) and (5.5), Eq. (5.6) shows that $-J^2$ is a projector to the tangent space of $\mathbf{CP}^{N-1}$.

2) *Metric*: For two Hermitian matrices $\mathcal{M}_1$ and $\mathcal{M}_2$ define

$$\mathcal{G}(\mathcal{M}_1, \mathcal{M}_2) := \mathrm{Tr}(-J^2(\mathcal{M}_1)\mathcal{M}_2) = -\mathrm{Tr}([P, \mathcal{M}_1][P, \mathcal{M}_2]). \tag{5.7}$$

This vanishes if any one of the arguments is a normal vector and on tangent vectors it agrees with the trace metric. It is the metric on $\mathbf{CP}^{N-1}$ induced from the trace metric for Hermitian matrices. One can show that $\mathcal{G}(J(\mathcal{M}_1), J(\mathcal{M}_2)) = \mathcal{G}(\mathcal{M}_1, \mathcal{M}_2)$.

3) *Symplectic structure*: For two matrices $\mathcal{M}_1$ and $\mathcal{M}_2$, define an antisymmetric two-form Ω by

$$\Omega(\mathcal{M}_1, \mathcal{M}_2) := \mathcal{G}(J(\mathcal{M}_1), \mathcal{M}_2) = -\mathrm{i}\,\mathrm{Tr}(\mathcal{P}[\mathcal{M}_1, \mathcal{M}_2]). \tag{5.8}$$

It vanishes if any of the arguments is normal to $\mathbf{CP}^{N-1}$. Thus, it is a two-form on $\mathbf{CP}^{N-1}$. It is in fact closed, as we shall show in Section 7.

One can combine $\mathcal{G}$ and Ω to obtain a tensor K on $\mathbf{CP}^{N-1}$,

$$K := \tfrac{1}{2}(\mathcal{G} + \mathrm{i}\Omega), \tag{5.9}$$

and it is a straightforward exercise to show that

$$K(\mathcal{M}_1, \mathcal{M}_2) = \mathrm{Tr}(\mathcal{P}\mathcal{M}_1\mathcal{M}_2) - \mathrm{Tr}(\mathcal{P}\mathcal{M}_1\mathcal{P}\mathcal{M}_2) = \mathrm{Tr}[\mathcal{P}\mathcal{M}_1(\mathbf{1} - \mathcal{P})\mathcal{M}_2]. \tag{5.10}$$

The construction of the Kähler structure described here also holds for other spaces of projection operators of a fixed rank, i.e., unitary Grassmannian manifolds. However, the fact that $\mathbf{CP}^{N-1}$ consists of rank 1 projection operators further simplifies the above equation to

$$K(\mathcal{M}_1, \mathcal{M}_2) := \mathrm{Tr}(\mathcal{P}\mathcal{M}_1\mathcal{M}_2) - \mathrm{Tr}(\mathcal{P}\mathcal{M}_1)\mathrm{Tr}(\mathcal{P}\mathcal{M}_2). \tag{5.11}$$

This form of K will be used crucially in the construction of fuzzy $\mathbf{CP}^{N-1}$ in the following section. In terms of the components with respect to the basis t_A (t_0 components all vanish) one finds

$$K_{AB} := K(t_A, t_B) = \frac{1}{N}\delta_{AB} + \frac{1}{\sqrt{2}}(d_{AB}^{C} + \mathrm{i} f_{AB}^{C})\xi_C - \xi_A\xi_B, \tag{5.12}$$

$$\mathcal{G}_{AB} = 2\,\mathrm{Re}\,K_{AB}, \qquad \Omega_{AB} = 2\,\mathrm{Im}\,K_{AB}, \qquad J_B^A = \delta^{AC}\Omega_{CB}. \tag{5.13}$$

Because of our normalisation of the matrices t_A, (4.2), the indices $A, B, \dots$ are raised and lowered with δ^{AB} and δ_{AB}, respectively. It is shown in the appendix that $P_B^A := \delta^{AC}\mathcal{G}_{CB}$ is a rank $2(N-1)$ projector and in fact $P = -J^2$. Alternatively, Eq. (5.7) will yield that result directly by splitting and combining traces containing the one-dimensional projector P as in (5.11). In future we shall not distinguish between $\mathcal{G}$ and P, nor between Ω and J, and shall write

$$K = \tfrac{1}{2}(P + \mathrm{i}J) \tag{5.14}$$

with

$$P_{AB} = \frac{2}{N}\delta_{AB} + \sqrt{2}(d_{AB}^{C}\xi_C) - 2\xi_A\xi_B, \tag{5.15}$$

and

$$J_{AB} = \sqrt{2} f_{AB}^{C}\xi_C. \tag{5.16}$$

In fact, as shown in the appendix, K itself is a rank $N-1$ projector—it can be interpreted as a projector from the redundant, global coordinates ξ_A to local (anti-)holomorphic coordinates on $\mathbf{CP}^{N-1}$. That K is a projector is obvious from (5.14), $J^2 = -P$ and $PJ = JP = J$.

6. Fuzzy complex projective spaces

We now turn to the construction of functions on $\mathbf{CP}^{N-1}$ and their $*$-product, generalising the construction given for $S_F^2 \cong \mathbf{CP}_F^1$ in [4]. While a non-commutative $*$-product on the continuum $\mathbf{CP}^{N-1}$ has been known for some time [21], we construct here a $*$-product on the fuzzy $\mathbf{CP}_F^{N-1}$, with a finite number of degrees of freedom.

In order to describe the harmonic expansion of functions on $\mathbf{CP}^{N-1}$ one only requires representations which are symmetric products of the fundamental representation of $SU(N)$, i.e., the $\mathbf{N}$ representation (or the complex conjugate $\bar{\mathbf{N}}$ representation). So the construction starts with an N-dimensional Hilbert space, $\mathcal{H}_1 := \mathbf{N} = \mathbf{C}^N$. To represent functions at the level L, we use as our Hilbert space , $\mathcal{H}_L$, which is the $d_L = (N-1+L)!/(N-1)!L!$-dimensional irreducible representation of $SU(N)$ obtained from the L-fold symmetric tensor product of $\mathcal{H}_1$. Associated with a point $\mathcal{P}$ in $\mathbf{CP}^{N-1}$ let us consider the L-fold tensor product of $\mathcal{P}$,

$$\mathcal{P}_L := \mathcal{P} \otimes \cdots \otimes \mathcal{P}. \tag{6.1}$$

Being an L-fold tensor product of the same operator, $\mathcal{P}_L$ is a well-defined operator on $\mathcal{H}_L$. Note that $\mathcal{P}_L$ is again a projection operator of rank 1. We will use this property of $\mathcal{P}_L$ later.

With each operator $\hat{F}$ on $\mathcal{H}_L$, we construct the corresponding function $F_L(\xi)$ on $\mathbf{CP}^{N-1}$ using the equivariant mapping prescription,

$$F_L(\xi) := \mathrm{Tr}(\mathcal{P}_L(\xi)\hat{F}). \tag{6.2}$$

In this way we define an injective mapping from operators $\hat{F}$ on $\mathcal{H}_L$ into functions F_L on $\mathbf{CP}^{N-1}$ (the injectivity is actually proved at the end of next section). The functions F_L are sufficient to reconstruct the operator $\hat{F}$. The target space of this mapping is derived in Section 7, it is what we denote by $\mathbf{CP}_F^{N-1}$ and is isomorphic to the space of $d_L \times d_L$ matrices.

A $*$-product on this space of functions is defined as

$$(F_L * G_L)(\xi) := \mathrm{Tr}(\mathcal{P}_L \hat{F}\hat{G}). \tag{6.3}$$

Associativity of the $*$-product is guaranteed by construction and derives from the associativity of matrix multiplication. Our aim is to derive an explicit, closed expression for the $*$-product (6.3) (or (3.3)), solely in terms of the functions F_L and G_L, and show that it reduces to the normal product of two functions in the limit $L \to \infty$.

At the level $L = 1$, the only functions allowed are functions linear in $\xi^{\hat{A}}$. This is because any Hermitian operator acting on the fundamental representation $\mathcal{H}_1$ of $SU(N)$, can be expanded in terms of $t_{\hat{A}}$. For $\hat{F} = F^{\hat{A}} t_{\hat{A}}$, the corresponding function $F_1(\xi)$ become

$$F_1(\xi) = F^{\hat{A}} \xi_{\hat{A}}. \tag{6.4}$$

In particular, $t_{\hat{A}}$ produces coordinate functions $\xi_{\hat{A}}$,

$$\xi_{\hat{A}} = \mathrm{Tr}(\mathcal{P} t_{\hat{A}}). \tag{6.5}$$

The $*$-product between coordinate functions, $\xi_{\hat{A}} * \xi_{\hat{B}} := \mathrm{Tr}(\mathcal{P} t_{\hat{A}} t_{\hat{B}})$ combined with (5.11) yields the following important relation:

$$\xi_{\hat{A}} * \xi_{\hat{B}} = \xi_{\hat{A}} \xi_{\hat{B}} + K_{\hat{A}\hat{B}}, \tag{6.6}$$

where $K_{\hat{A}\hat{B}}$ is the Hermitian structure. Note that $K_{0\hat{A}}$ vanishes for all $\hat{A}$.

For any finite L, functions and their $*$-product are constructed using Hermitian operators on $\mathcal{H}_L$ according to the prescriptions (6.2) and (6.3). Given two operators $\hat{F}$ and $\hat{G}$ write them in the following form,

$$\hat{F} = F_{\hat{A}_1 \cdots \hat{A}_L} t^{\hat{A}_1} \otimes \cdots \otimes t^{\hat{A}_L}, \qquad \hat{G} = G_{\hat{A}_1 \cdots \hat{A}_L} t^{\hat{A}_1} \otimes \cdots \otimes t^{\hat{A}_L}, \tag{6.7}$$

where the coefficient tensors are totally symmetric. Of course, for a given operator on $\mathcal{H}_L$ there is no unique expression of the above form. In fact, a choice of symmetric tensor corresponds to a particular extension of the operator to the whole tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_1 \cdots \otimes \mathcal{H}_1$. This ambiguity will eventually disappear because the construction of functions and their $*$-product depend only on operators acting on $\mathcal{H}_L$. The functions corresponding to (6.7) are

$$F_L(\xi) = F_{\hat{A}_1 \cdots \hat{A}_L} \xi^{\hat{A}_1} \cdots \xi^{\hat{A}_L}, \qquad G_L(\xi) = G_{\hat{A}_1 \cdots \hat{A}_L} \xi^{\hat{A}_1} \cdots \xi^{\hat{A}_L}, \tag{6.8}$$

and their $*$-product becomes

$$(F_L * G_L)(\xi) = F_{\hat{A}_1 \cdots \hat{A}_L} G_{\hat{B}_1 \cdots \hat{B}_L} (\xi^{\hat{A}_1} * \xi^{\hat{B}_1}) \cdots (\xi^{\hat{A}_L} * \xi^{\hat{B}_L}). \tag{6.9}$$

Since $\xi^0 = 1/\sqrt{N}$ is a constant, all functions can be considered as polynomials in just ξ^A of degree $\leq L$.

Now, in order to express this in the final form, we first substitute the relation (6.6) into the above equation and expand it in powers of $K^{\hat{A}\hat{B}}$ to get

$$\begin{aligned}(F_L * G_L)(\xi) = F_L(\xi) G_L(\xi) + \sum_{l=1}^{L} \frac{L!}{(L-l)!l!} F_{\hat{A}_1 \cdots \hat{A}_l \hat{A}_{l+1} \cdots \hat{A}_L} \xi^{\hat{A}_{l+1}} \cdots \xi^{\hat{A}_L} \\ \times G_{\hat{B}_1 \cdots \hat{B}_l \hat{B}_{l+1} \cdots \hat{B}_L} \xi^{\hat{B}_{l+1}} \cdots \xi^{\hat{B}_L} K^{\hat{A}_1 \hat{B}_1} \cdots K^{\hat{A}_l \hat{B}_l},\end{aligned} \tag{6.10}$$

where the first term is the ordinary commutative product, and will be integrated into the sum as the $l = 0$ term for convenience. Finally, using the relation

$$\partial_{\hat{A}_1} \cdots \partial_{\hat{A}_l} F_L(\xi) = \frac{L!}{(L-l)!} F_{\hat{A}_1 \cdots \hat{A}_l \hat{A}_{l+1} \cdots \hat{A}_L} \xi^{\hat{A}_{l+1}} \cdots \xi^{\hat{A}_L}, \tag{6.11}$$

and the fact that $K^{0\hat{A}} = 0$, we get

$$(F_L * G_L)(\xi) = \sum_{l=0}^{L} \frac{(L-l)!}{L!l!} [\partial_{A_1} \cdots \partial_{A_l} F_L(\xi)] K^{A_1 B_1} \cdots K^{A_l B_l} [\partial_{B_1} \cdots \partial_{B_l} G_L(\xi)]. \tag{6.12}$$

Note again that in arriving at (6.12) we have extended functions and derivatives to outside $\mathbf{CP}^{N-1}$ and finally evaluated the result on $\mathbf{CP}^{N-1}$. However, this extension should be regarded as a convenient way of calculation because the final expression involves functions on $\mathbf{CP}^{N-1}$ and derivatives along $\mathbf{CP}^{N-1}$ only, as we will explicitly show below.

Eq. (6.12) is one of the central results of this paper and generalises the result for S^2 derived in [4]. Only the $l = 0$ term survives in the limit $L \to \infty$, which shows that the $*$-product reduces to ordinary multiplication of functions in the continuum limit, with corrections being of order $1/L$. Note that the limit should be taken with all functions fixed.

As mentioned earlier, and proven in the appendix, the matrix K_{AB} is a projector. In fact the derivatives in (6.12), which are flat in $\mathbf{R}^{N^2-1}$ are being projected onto the holomorphic tangent space of $\mathbf{CP}^{N-1}$ and are actually covariant derivatives there. Note that, since K is Hermitian, it gives a holomorphic derivative when acting to the right, as in $K^{AB}(\partial_B F)$, but an anti-holomorphic derivative when acting to the left, as in $(\partial_B F)K^{BA} = \bar{K}^{AB}(\partial_B F)$, where the bar represents complex conjugation. Thus, if our algebra of functions permitted us to construct holomorphic or anti-holomorphic functions, the $*$-product of a (anti-) holomorphic function with another (anti-) holomorphic function would always reduce to the ordinary product. More generally the $*$-product, $F_L * G_L$, is an ordinary product if G is anti-holomorphic regardless of the form of F or, conversely, if F is holomorphic regardless of the form of G. Another point to note is that the complex structure is reversed, $J \to -J$, if the original Hilbert space is identified with the complex conjugate fundamental representation $\bar{\mathbf{N}}$ rather than the $\mathbf{N}$.

The structure here is perhaps most clearly understood by looking at the simplest case, $N = 2$. Then $P_{AB} = \delta_{AB} - 2\xi_A\xi_B$ and $J_{AB} = \sqrt{2}\epsilon_{ABC}\xi^C$. The constraints imply that $\xi_A\xi^A = 1/2$ and so define a unit vector in $\mathbf{R}^3$, $n_A = \sqrt{2}\xi_A$, so that $P_{AB} = \delta_{AB} - n_A n_B$ and $J_{AB} = \epsilon_{ABC}n^C$. Clearly, $P = -J^2$ and P is a projector from $\mathbf{R}^3$ onto the unit sphere while J, when restricted to $\mathbf{n}.\mathbf{n} = 1$, represents the complex structure on $\mathbf{CP}^1$. In view of the identity $J^3 = -J$, the combination $K = (P + \mathrm{i}J)/2$ is a rank 1 projector onto a complex holomorphic coordinate on $\mathbf{CP}^1$ ($JK = -\mathrm{i}K$). This interpretation survives to higher N also and gives a geometric interpretation of the $*$-product (6.12).

In a standard geometrical construction a covariant derivative on a curved space can be obtained by embedding the space in a flat Euclidean space of higher dimension and projecting the ordinary derivative in the Euclidean space onto the tangent space of the curved manifold. When the Euclidean derivatives are restricted to act on tensors already projected to the tangent space of the curved manifold, the projected flat derivative is a covariant derivative. There is a simplification in the construction here, because the projector K_{AB} satisfies [17]

$$K^{AB}K^{CD}\partial_B K_{DE} = 0, \tag{6.13}$$

which implies that

$$K^{AB}K^{CD}\partial_B(K^E_D\partial_E F) = K^{AB}K^{CD}\partial_B\partial_D F, \tag{6.14}$$

since $K^2 = K$, with an obvious generalisation to derivatives acting on higher rank tensors. This identity can be proven using the last form of K_{AB} in (5.10), $K_{AB} = \mathrm{Tr}[\mathcal{P}t_A(\mathbf{1} - \mathcal{P})t_B]$, and completeness of the matrices t_A. An alternative, more detailed proof, is given in Appendix B.

So, defining $\nabla_A := K_A^B \partial_B$ and $\bar{\nabla}_A := \bar{K}_A^B \partial_B$ and using (6.14) and its generalisation to convert the successive partial derivative to covariant derivatives in (6.12), the $*$-product is

$$(F_L * G_L)(\xi) = \sum_{l=0}^{L} \frac{(L-l)!}{L!l!} [\bar{\nabla}_{A_1} \cdots \bar{\nabla}_{A_l} F_L(\xi)] K^{A_1 B_1} \cdots K^{A_l B_l} \times [\nabla_{B_1} \cdots \nabla_{B_l} G_L(\xi)]. \tag{6.15}$$

Converting from global coordinates, ξ^A with $A = 1, \ldots, N^2 - 1$, to local holomorphic coordinates, z^i with $i = 1, \ldots, N-1$ and $z^{\bar{i}} := \bar{z}^i$ we have the correspondences

$$K_{AB} \to \tfrac{1}{2}(\mathcal{G}_{i\bar{j}} + \mathrm{i}\Omega_{i\bar{j}}) = \mathrm{i}\Omega_{i\bar{j}}, \qquad K^{AB} \to \tfrac{1}{2}(\mathcal{G}^{\bar{j}i} + \mathrm{i}\Omega^{\bar{j}i}) = \mathrm{i}\Omega^{\bar{j}i}, \tag{6.16}$$

where $\mathcal{G}_{i\bar{j}}$ is the Fubini–Study metric and $\Omega_{i\bar{j}}$ the Kähler 2-form, with $\mathcal{G}_{i\bar{j}} = \mathcal{G}_{\bar{j}i} = \mathrm{i}\Omega_{i\bar{j}} = -\mathrm{i}\Omega_{\bar{j}i}$, and $\Omega^{\bar{j}i} = \mathcal{G}^{\bar{j}n}\mathcal{G}^{i\bar{m}}\Omega_{n\bar{m}}$. Eq. (6.15) in local coordinates takes the form

$$(F_L * G_L)(z, \bar{z}) = \sum_{l=0}^{L} \frac{(L-l)!}{L!l!} [\nabla_{\bar{j}_1} \cdots \nabla_{\bar{j}_l} F_L(z, \bar{z})](\mathrm{i}\Omega^{\bar{j}_1 i_1}) \cdots (\mathrm{i}\Omega^{\bar{j}_l i_l}) \times [\nabla_{i_1} \cdots \nabla_{i_l} G_L(z, \bar{z})], \tag{6.17}$$

where ∇_i is the covariant derivative.

7. Fuzzy derivatives

The star product defined here can be used for more than just multiplying functions on the fuzzy $\mathbf{CP}_F^{N-1}$, it can also be used to define derivatives on the discrete fuzzy spaces. In the continuum the vector fields on $\mathbf{CP}^{N-1}$ generating $SU(N)$ can be expressed as

$$\mathcal{L}_A = -\mathrm{i} f_{AB}^C \xi^B \partial_C = \mathrm{i}\frac{1}{\sqrt{2}} J_A^C \partial_C. \tag{7.1}$$

It is easily verified that

$$[\mathcal{L}_A, \mathcal{L}_B] = \mathrm{i} f_{AB}^C \mathcal{L}_C. \tag{7.2}$$

The corresponding action of a generator L_A on the Hilbert space $\mathcal{H}_L$ is obtained from exponentiating the generator, that is by considering $D_L(\eta) = \mathrm{e}^{\mathrm{i}\eta^A L_A}$:

$$\mathrm{Tr}[\mathcal{P}_L(\xi) D_L(\eta^{-1}) \hat{F} D_L(\eta)] = \mathrm{Tr}[\mathcal{P}_L(\xi_0) D_L(g^{-1}\eta^{-1}) \hat{F} D_L(\eta g)]. \tag{7.3}$$

Infinitesimally, with η_A small and $D_L^{-1}(\eta) \approx 1 - \mathrm{i}\eta^A L_A$,

$$\mathrm{Tr}[\mathcal{P}_L(\xi)(1 - \mathrm{i}\eta^A L_A)] = \left\{\mathrm{Tr}\left[\mathcal{P}_1(\xi)\left(1 - \mathrm{i}\eta^A\left(\frac{t_A}{\sqrt{2}}\right)\right)\right]\right\}^L \approx 1 - \frac{\mathrm{i}L}{\sqrt{2}}\eta^A \xi_A. \tag{7.4}$$

So $\mathrm{Tr}[\mathcal{P}_L(\xi) L_A] = (L/\sqrt{2})\xi_A$. (The generators (4.3) in the fundamental representation were normalised so that $[t_A/\sqrt{2}, t_B/\sqrt{2}] = \mathrm{i} f_{AB}^C (t_C/\sqrt{2})$.)

Now the derivative of a function in the continuum, $\mathcal{L}_A F(\xi)$, can be taken over to the fuzzy case as

$$(\mathcal{L}_A F_L)(\xi) := \mathrm{Tr}\{\mathcal{P}_L(\xi)[L_A, \hat{F}]\} = \frac{L}{\sqrt{2}}(\xi_A * F_L - F_L * \xi_A). \tag{7.5}$$

Using the $*$-product (6.12) this is

$$(\mathcal{L}_A F_L)(\xi) = \frac{1}{\sqrt{2}}(K^{AB}\partial_B F_L - (\partial_B F_L)K^{BA}) = \frac{\mathrm{i}}{\sqrt{2}} J^{AB}\partial_B F_L, \tag{7.6}$$

and this shows that the definition (7.5) is consistent with (7.1). The main point here is that derivatives on functions in the continuum restrict to derivatives at finite L which can be represented as commutators in the matrix algebra,

$$(\mathcal{L}_A F_L)(\xi)) = \mathrm{Tr}\{\mathcal{P}_L(\xi)[L_A, \hat{F}]\}. \tag{7.7}$$

This formula can now be used to prove that the symplectic form, Ω, defined in (5.8) is closed. Let Lie_X denote the Lie derivative along the vector field X. Then, in the formula for the exterior derivative of a 2-form acting on three tangent vectors, $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$,

$$\begin{aligned} \mathrm{d}\Omega(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = {} & \mathrm{Lie}_{\mathbf{X}}\, \Omega(\mathbf{Y}, \mathbf{Z}) + \mathrm{Lie}_{\mathbf{Y}}\, \Omega(\mathbf{Z}, \mathbf{X}) + \mathrm{Lie}_{\mathbf{Z}}\, \Omega(\mathbf{X}, \mathbf{Y}) - \Omega([\mathbf{X}, \mathbf{Y}], \mathbf{Z}) \\ & - \Omega([\mathbf{Y}, \mathbf{Z}], \mathbf{X}) - \Omega([\mathbf{Z}, \mathbf{X}], \mathbf{Y}), \end{aligned} \tag{7.8}$$

we represent all tangent vector fields by matrices as in (7.7) (any tangent vector can be written as a linear combination of the $\mathcal{L}_A$ at each ξ) and conclude that $\mathrm{d}\Omega = 0$ by the Jacobi identity.

At this point, it is possible to derive simply the target space of the mapping (6.2) from operators $\hat{F}$ on $\mathcal{H}_L$ to functions $F_L(\xi)$ on $\mathbf{CP}^{N-1}$. Since the derivations $[\cdot, L_A]$ in $\mathcal{H}_L$ are sent exactly to the derivations $\mathcal{L}_A$ in $\mathbf{CP}^{N-1}$ by the mapping, the second order Casimir in the adjoint action in $\mathcal{H}_L$ is mapped to the Laplacian in $\mathbf{CP}^{N-1}$, and the commutator actions of the Cartan sub-algebra operators are sent to their equivalent derivations in $\mathbf{CP}^{N-1}$. This means that the normalised simultaneous eigenvectors of all these Cartan operators in $\mathcal{H}_L$ are mapped to simultaneous eigenfunctions of all the corresponding derivation operators in $\mathbf{CP}^{N-1}$ with the same eigenvalues. Denoting the irreducible tensor operators which are eigenvectors of the Cartan operators by $\hat{T}^{\mathbf{J}}_{\mathbf{M}}$, with $\mathbf{J}$ a multiple index labelling the representation and $\mathbf{M}$ a multi-index labelling the weights, we find that $\hat{T}^{\mathbf{J}}_{\mathbf{M}}$ are mapped to $c^{\mathbf{J}}(L)Y^{\mathbf{J}}_{\mathbf{M}}$, $Y^{\mathbf{J}}_{\mathbf{M}}$ being spherical harmonics, the analogues of Y^l_M for $SU(2)$. The constants $c^{\mathbf{J}}(L)$ can easily be calculated and are all non-zero, which implies the injectivity of the mapping F_L assumed in (6.2). Thus, the target of the mapping is just the space generated by the eigenfunctions $Y^{\mathbf{J}}_{\mathbf{M}}$ of the Laplacian which are images of the $\hat{T}^{\mathbf{J}}_{\mathbf{M}}$, with $\mathbf{J}$ running over all $SU(N)$ irreducible representations in the $d_L \times d_L$ reducible representation that contain $U(N)$ singlets. For example $\mathbf{CP}^1 \cong S^2 \cong SU(2)/U(1)$ requires L-fold symmetric representations with $d_L = (L+1)$ and the $(L+1)\times(L+1)$ reducible representation decomposes into irreducible representations as $\mathbf{1}+\mathbf{3}+\cdots+(\mathbf{2L}+\mathbf{1})$. There is only one Casimir for $SU(2)$, so $\mathbf{J}$ is just the integer l of the associated irreducible representation and M is the magnetic quantum number. The $\hat{T}^l_M$, $l = 0, \ldots, L$, are a basis for all $(L+1)\times(L+1)$ matrices and Y^l_M are the usual spherical harmonics.

8. Conclusions

The central result of this paper is Eq. (6.12), which gives the explicit construction of an associative $*$-product on the fuzzy $\mathbf{CP}_F^{N-1}$ between two functions $F_L = \mathrm{Tr}\{\mathcal{P}_L \hat{F}\}$ and $G_L = \mathrm{Tr}\{\mathcal{P}_L \hat{G}\}$,

$$(F_L * G_L)(\xi) = F_L(\xi)G_L(\xi) + \sum_{l=1}^{L} \frac{(L-l)!}{L!l!} \times [\partial_{A_1} \cdots \partial_{A_l} F_L(\xi)] K^{A_1 B_1} \cdots K^{A_l B_l} [\partial_{B_1} \cdots \partial_{B_l} G_L(\xi)].$$

This expression is written in terms of an over-complete set of coordinates ξ^A in $\mathbf{R}^{N^2-1}$, with constraints (4.5). The projector $K = (P + \mathrm{i}J)/2$ in Eq. (5.14) is defined by

$$P_{AB} = \frac{2}{N}\delta_{AB} + \sqrt{2}(d_{AB}^C \xi_C) - 2\xi_A \xi_B,$$

and

$$J_{AB} = \sqrt{2} f_{AB}^C \xi_C.$$

$P = -J^2$ is itself a projector mapping $\mathbf{R}^{N^2-1}$ onto the tangent plane of $\mathbf{CP}^{N-1}$ at ξ^A.

P and J are essentially the components of the usual Hermitian structure on $\mathbf{CP}^{N-1}$ obtained by embedding it in the space of Hermitian matrices $\mathbf{R}^{N^2}$. The latter is encapsulated in the three equations (5.5), (5.7) and (5.8):

$$J(\mathcal{M}) := \mathrm{i}[\mathcal{P}, \mathcal{M}], \qquad \mathcal{G}(\mathcal{M}_1, \mathcal{M}_2) := \mathrm{Tr}(-J^2(\mathcal{M}_1), \mathcal{M}_2),$$

and

$$\Omega(\mathcal{M}_1, \mathcal{M}_2) := \mathrm{Tr}(\mathcal{M}_1 J(\mathcal{M}_2)) = -\mathrm{i}\,\mathrm{Tr}(\mathcal{P}[\mathcal{M}_1, \mathcal{M}_2]),$$

describing the complex structure, the Fubini–Study metric and the symplectic structure on $\mathbf{CP}^{N-1}$ respectively. In our normalisation convention $P = \mathcal{G}$. Expressed in local holomorphic coordinates z^i, $i = 1, \ldots, N-1$, rather than the global coordinates, ξ^A, this $*$-product is (5.17),

$$(F_L * G_L)(z, \bar{z}) = F_L(z, \bar{z})G_L(z, \bar{z}) + \sum_{l=1}^{L} \frac{(L-l)!}{L!l!} \times [\nabla_{\bar{j}_1} \cdots \nabla_{\bar{j}_l} F_L(z, \bar{z})](\mathrm{i}\Omega^{\bar{j}_1 i_1}) \cdots (\mathrm{i}\Omega^{\bar{j}_l i_l})[\nabla_{i_1} \cdots \nabla_{i_l} G_L(z, \bar{z})].$$

The $*$-star product reduces to the ordinary commutative product on the continuous $\mathbf{CP}^{N-1}$ in the $L \to \infty$ limit for fixed F_L and G_L [20].

Note also the important expression for the derivative of a function on the fuzzy $\mathbf{CP}_F^{N-1}$ as a commutator (7.7), which appears naturally in this construction

$$\mathcal{L}_A F_L = \mathrm{Tr}\{\mathcal{P}_L [L_A, \hat{F}]\}.$$

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Appendix A

In this appendix we derive some essential properties of the matrix $K = (K_{AB})$ used in the definition of the $*$-product (6.12). First we show that K is a projector, with rank $N-1$. To this end break K into real and imaginary parts as in the text, $K = \frac{1}{2}(P + \mathrm{i}J)$ with

$$P_{AB} := \frac{2}{N}\delta_{AB} - 2\xi_A\xi_B + \sqrt{2}\mathcal{S}_{AB}, \tag{A.1}$$

and

$$J_{AB} := \sqrt{2}\mathcal{A}_{AB} \tag{A.2}$$

with symmetric matrix $\mathcal{S}_{AB} := d_{AB}^{C}\xi_C$ and the anti-symmetric matrix $\mathcal{A}_{AB} := f_{AB}^{C}\xi_C$ (all indices are raised and lowered here using δ_{AB}). It is shown in the text that J corresponds to the complex structure on $\mathbf{CP}^{N-1}$, and we show here that $-J^2$ is a projector of rank $2(N-1)$, with $PJ = JP = J$ and finally $J^2 = -P$, which implies in particular that P itself is also a projector.

i) *K is a projector with rank $N-1$.* To see this observe that

$$\mathrm{Tr}(t_At_Bt_Ct_D) = \frac{1}{N}\delta_{AB}\delta_{CD} + \frac{1}{2}(d_{AB}^{E} + \mathrm{i}f_{AB}^{E})(d_{ECD} + \mathrm{i}f_{ECD}). \tag{A.3}$$

Now contracting this with $\xi^C\xi^D$ and using cyclic symmetry of the trace and the constraints (4.5) yields the two identities:

$$\mathcal{S}_{AB}^2 - \mathcal{A}_{AB}^2 = \frac{2(N-1)}{N^2}\delta_{AB} - \frac{2}{N}\xi^A\xi^B + \frac{\sqrt{2}(N-2)}{N}\mathcal{S}_{AB}, \tag{A.4}$$

and

$$(\mathcal{S}\mathcal{A} + \mathcal{A}\mathcal{S})_{AB} = \frac{\sqrt{2}(N-2)}{N}\mathcal{A}_{AB}. \tag{A.5}$$

From these it follows easily that $K^2 = K$. Since the constraints also dictate that $\mathrm{tr}(K) = N-1$ (tr here means trace over the adjoint representation of $SU(N)$, so $\delta_A^A = N^2 - 1$), K is a projector onto an $N-1$ dimensional subspace of $\mathbf{R}^{N^2-1}$.

ii) *J^2 is a projector and $J^3 = -J$*. In the text the complex structure was denoted by J, and we can identify that with the symplectic structure when the normalisation is such that indices are raised and lowered with δ_{AB}. For completeness we give here an alternative derivation. First we show that $J^3 = -J$ and $\mathrm{tr}(-J^2) = 2(N-1)$. By definition $J_{AB} := \sqrt{2}\mathcal{A}_{AB} = \sqrt{2} f_{ABC}\xi^C$, so $J_{AB}t^B = \mathrm{i}[t_A, \xi]$, where $\boldsymbol{\xi} = \xi^A t_A$. The constraints (4.5) imply

$$\xi^2 = \left(\frac{N-1}{N^2}\right)\mathbf{1} + \left(\frac{N-2}{N}\right)\xi. \tag{A.6}$$

Using the commutation relations for t_A gives

$$[[[t_A, \xi], \xi], \xi] = \mathrm{i}(J^3)_{AB}t^B, \tag{A.7}$$

while expanding the commutators on the left hand side explicitly, and using (A.6), gives

$$[[[t_A, \xi], \xi], \xi] = -\mathrm{i}J_{AB}t^B, \tag{A.8}$$

from which we conclude that $J^3 = -J$. This means that $-J^2$ is a projector since $(-J^2)^2 = (-J^2)$ and the definition of J, (A.2), together with the constraints (4.5) and the standard normalisation $f_{ACD}f_{BCD} = N\delta_{AB}$, show that $\mathrm{tr}(-J^2) = 2(N-1) = \dim \mathbf{CP}^{N-1}$.

iii) *J commutes with P and $PJ = J$*. Since d_{ABC} is an invariant tensor we have

$$f_{AB}^{H}d_{HCD} + f_{AC}^{H}d_{BHD} + f_{AD}^{H}d_{BCH} = 0,$$

and contracting this with $\xi^A\xi^B$ shows that $\mathcal{S}$ commutes with $\mathcal{A}$, since f_{ABC} is totally antisymmetric. The latter also means that J annihilates ξ, so that J commutes with P. Since $K^2 = K$ we have $J = (PJ + JP)/2$, and hence $PJ = J$.

iv) $P = -J^2$. The real part of $K^2 = K$ implies that $P^2 - J^2 = 2P$. Since P commutes with J they are simultaneously diagonalisable and because $-J^2$ is a projector, its eigenvalues are all 0 or 1. So the eigenvalues of P are 1 when the eigenvalue of $-J^2$ is 1, and either 0 or 2 when the eigenvalue of $-J^2$ is 0. Calling p the number of eigenvalues 2 in P, we have

$$\mathrm{tr}(P) = \mathrm{tr}(-J^2) + 2p, \tag{A.9}$$

while, directly from the definition of P (A.1) and the constraint equations (4.5), one finds

$$\mathrm{tr}(P) = 2(N-1) = \mathrm{tr}(-J^2), \tag{A.10}$$

which implies that $p = 0$. Thus we have $P = -J^2$, with P a projector of rank $2(N-1)$. Note that this implies that K annihilates the coordinates $K_{AB}\xi^B = 0$, since J does, which is easily checked using (A.2).

Appendix B

In this appendix we give an alternative, more detailed, proof of the identity (6.13),

$$K^{AB}K^{CD}\partial_B K_{DE} = 0. \tag{6.13}$$

Denoting the generators of $SU(N)$ in the adjoint representation by $(\theta_A)_{BC} = -\mathrm{i} f_{ABC}$, with commutation relations $[\theta_A, \theta_B] = \mathrm{i} f_{ABC}\theta_C$, we have $J = \mathrm{i}\sqrt{2}\theta_A \xi^A$ and

$$J_{AB}\partial_B J = \sqrt{2}\mathrm{i}[\theta_A, J], \qquad J_{AB}\theta_B = [\theta_A, J]. \tag{B.1}$$

Using these commutators it is straightforward to show that

$$K_{AB}\partial_B K = \frac{1}{\sqrt{2}}(\mathbf{1} + \mathrm{i}J)_{AB}[K, \theta_B] = \frac{1}{\sqrt{2}}([K, [\theta_A, \mathrm{i}J]] + [K, \theta_A]). \tag{B.2}$$

Now, since $K^2 = K$ we have $K(\mathrm{d}K) + (\mathrm{d}K)K = \mathrm{d}K$ from which

$$K(\mathrm{d}K) = \mathrm{d}K(1 - K). \tag{B.3}$$

The eigenvalues of $\mathrm{i}J$ are ± 1 (each with multiplicity $(N-1)$) and 0 (with multiplicity $(N-1)^2$). We can thus choose a basis where

$$\mathrm{i}J = \begin{pmatrix} \mathbf{1}_{(N-1)} & & \\ & \mathbf{0}_{(N-1)^2} & \\ & & -\mathbf{1}_{(N-1)} \end{pmatrix}, \tag{B.4}$$

and

$$K = \begin{pmatrix} \mathbf{1}_{(N-1)} & & \\ & \mathbf{0}_{(N-1)^2} & \\ & & \mathbf{0}_{(N-1)} \end{pmatrix} = \begin{pmatrix} \mathbf{1}_{(N-1)} & \\ & \mathbf{0}_{N(N-1)} \end{pmatrix}, \tag{B.5}$$

where, for example, $\mathbf{1}_{(N-1)}$ is the $(N-1)\times(N-1)$ identity matrix and $\mathbf{0}_{(N-1)}$ the $(N-1)\times(N-1)$ square matrix with all entries zeros. In terms of the 2×2 block structure of the second form of K above, we write

$$\mathrm{d}K = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}.$$

Eq. (B.3) then shows that

$$K\,\mathrm{d}K = \begin{pmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

so we only need examine $< 1|K\,\mathrm{d}K|0 >$ and $< 1|K\,\mathrm{d}K|-1 >$, where $\mathrm{i}J|n >= n|n >$. Now from (B.2)

$$KK_{AB}\partial_B K = \frac{1}{\sqrt{2}}K([[\mathrm{i}J, \theta_A], K] + [K, \theta_A]), \tag{B.6}$$

and, since $K|1\rangle = |1\rangle$, $K|0\rangle = K|-1\rangle = 0$, we deduce that

$$\langle 1|KK_{AB}\partial_B K|0\rangle = \frac{1}{\sqrt{2}}\langle 1|[\theta_A, \mathrm{i}J] + \theta_A|0\rangle = 0, \tag{B.7}$$

$$\langle 1|KK_{AB}\partial_B K|-1\rangle = \frac{1}{\sqrt{2}}\langle 1|([\theta_A, \mathrm{i}J]+\theta_A)|-1\rangle = -\frac{1}{\sqrt{2}}\langle 1|\theta_A|-1\rangle. \tag{B.8}$$

The last expression vanishes, because θ_A does not connect $|1\rangle$ and $|-1\rangle$, and (6.13) follows.

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